Workforce Demographics and Technology Adoption

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Introduction

- Population Aging is transforming advanced economies.
  - U.S.: labor force growth ↓ from 2% in 70s to 0.8% today, and to 0.3% by 2060.

Q. What is the impact of population aging on aggregate labor productivity?

- Standard View
  - Scale effect reduces output, productivity, and firm creation.
    Hopenhayn et al. [2022],Maestas et al. [2023],Bloom et al. [2020],Karahan et al. [2024]
- Our Focus: Composition shifts from younger to older workers.
  - Old: accumulates more experience on the job $\Longrightarrow$ ↑ Productivity → Standard
  - Young: better endowed with up-to-date skills $\Longrightarrow$ ↑ Technology Adoption → New
- Effects on aggregate productivity ex-ante ambiguous + multiple channels at play.

Today's Talk

Empirics: young workers have a comparative advantage in new technologies.

1. Firms with Younger workforce: $\uparrow$ Prob. Adopt Tech.
2. Young workers: $\uparrow$ Prob. tech-related tasks + Productivity Gains from Adoption.
3. Adoption tilts workforce toward young workers.

Theory: firm dynamics + labor market frictions + multidimensional skills.

- Young workers: better at tech tasks, Old workers: better at production tasks.
- Advanced technologies are intensive in tech tasks.
- **Q.** Effect of **population aging** on **productivity** through:
    1. Endogenous Technology Adoption;
    2. Allocation of Workers to Firms.

Outline

1. Empirical Analysis

2. The Model

3. Productivity and Misallocation

Data

- LPP-ADIAB: Survey of ICT adoption.
  - Employer + Employees: reports on tech usage and its effects (2015, 2019, 2021).
- BHP (1993-2021): Admin Data on firms' size and workforce composition.
- IAB-BP (1993-2021): Annual representative survey of establishments.
  - Value Added, Investment, On-the-job training, Digitalization Info.
- SIAB (1985-2021): Admin Data on employment histories.
  - Wage, Occupations, Unemployment Spells, Education Level, Demographics.

LPP-ADIAB Overview

- Sample: $\approx$ 750 establishments per wave.
  - Representative of firms > 50 employees.
  - Representative survey of employees are surveyed (7 per firm avg.)
- Advanced technologies surveyed
  - Digital: e.g., IoT, Big Data Analytics, AI, Virtual Reality.
  - Automation: e.g., Robots, Drones, Additive Manufacturing.
- Usage Options (for each tech.)
  - Use < 2 years ago $\rightarrow$ Adoption Event at $t-2$.
  - Use > 2 years ago $\rightarrow$ Control/Exclude.
  - Not Use.

Workforce Composition and Tech Adoption

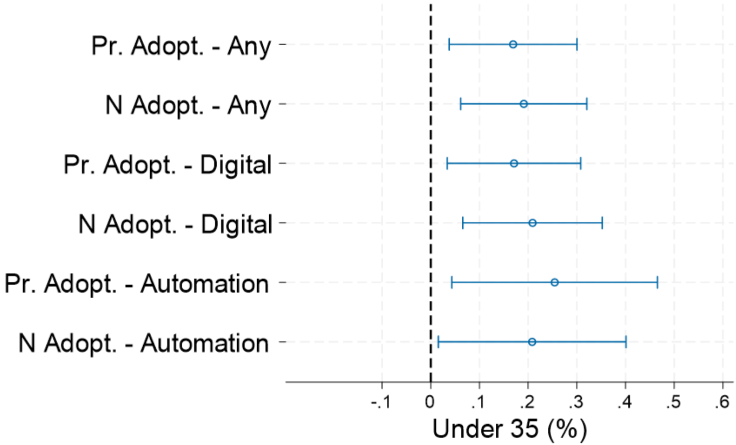
Q. Are firms with a younger workforce more likely to adopt advanced tech?

Empirical Specification

$$\mathbb{I}(\text{Adopt})_t = \alpha + \beta \text{ Under 35 } (\%)_{t-1} + \Gamma X_{t-1} + \varepsilon_t$$

- Baseline Controls: Firm Size, Firm age, Avg. Wage, College (%), Hiring Rate, Type of Owner.
- Fixed Effects: Sector 3-digit, State, Year, # Tech already used.

Young Workforce Predicts Higher Adoption



- Young Workforce $\implies$ Extensive ($\uparrow$ Pr. Adopt) + Intensive margin (# Techs)

Tasks and Tech Effect on Workers

Q1. Are younger workers more likely to carry out tech-related tasks?

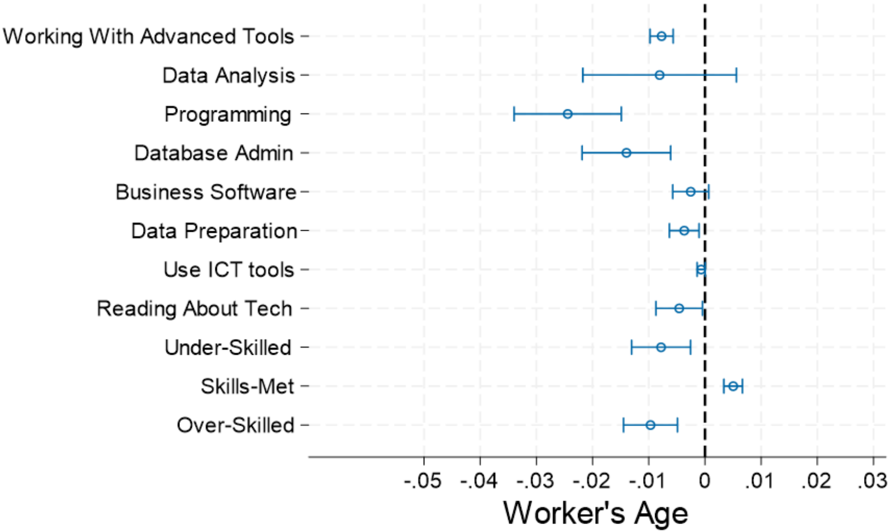
Q2. How do new technologies affect workers' tasks and productivity?

Empirical Specification

$$\mathbb{I}(\mathrm{Y}_{it}) = \alpha + \beta \ \mathrm{Age}_{it} + \Gamma X_{it} + \underbrace{\gamma_{f(it)}}_{\text{Firm FE}} + \varepsilon_{f(it),t}$$

- Outcomes (Y_{it})
  - Task-Related Questions (All Sample)
  - Impact of Tech (Only in Adopting Firms)
- Baseline Controls: Wage, Occupation Tenure, Firm Age.
- Fixed Effects: Firm, Year, Task Complexity, College, Occupation (3-digit)

Young perform more Tech-Related Tasks



Young report Productivity Gains for Tech Adoption

Event Study Around Adoption

Q. How does firms' employment and workforce composition change after adoption?

Empirical Strategy

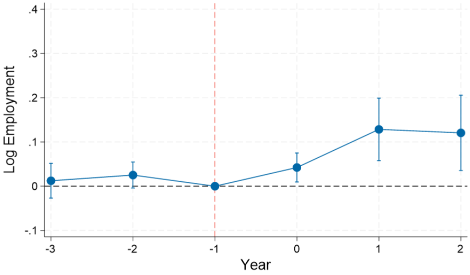
- Compare dynamic outcomes of adopting vs non-adopting establishments.

Empirical Specification

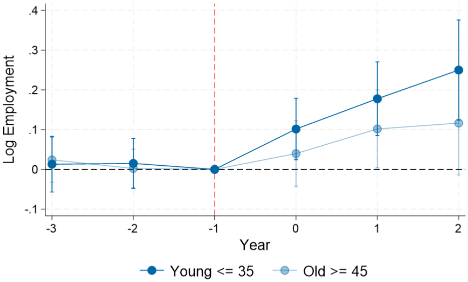
$$y_{ft} = \alpha_f + \gamma_t + \sum_k \underbrace{\theta_k D_{ft}^k}_{\text{Non-Adopters}} + \sum_k \underbrace{\beta_k (D_{ft}^k \times \text{Tech Adoption}_f)}_{\text{Adopters}} + \varepsilon_{ft},$$

- Outcomes (y_{ft}): Log-Employment, Empl. Composition, Empl. Shares, Wages
- Fixed Effects: Establishment, Sector 3-digit, State, Year.

Event Study - Employment



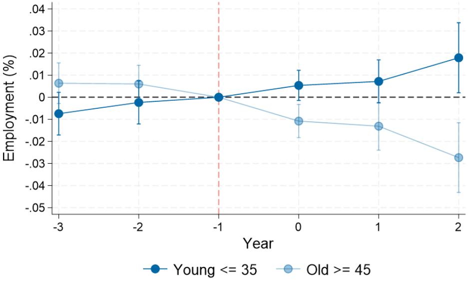
(a) Log Employment



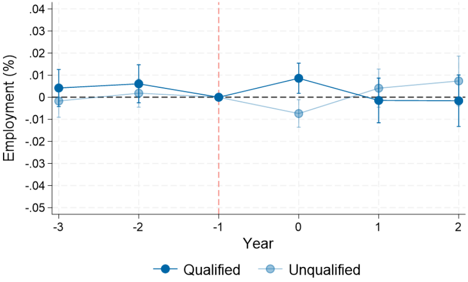
(b) Age Heterogeneity

- Firms expand employment primarily by hiring young.

Event Study - Employment Share



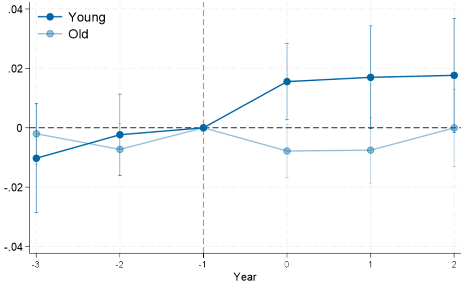
(a) Age Heterogeneity



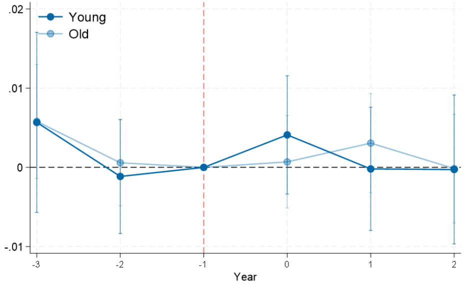
(b) College Heterogeneity

- Workforce tilts toward young workers.
- Age shift not driven by education composition.

Event Study - Incumbent Workers



(a) Log-Wage



(b) Days Worked

- Young Incumbents enjoy higher earnings $\Longrightarrow$ Productivity Gains.
- No observable trend in days worked.

Outline

1. Empirical Analysis

2. The Model

3. Productivity and Misallocation

Overview of the Model

- Model Ingredients
  1. Firm Dynamics + Endogenous Technology Adoption.
  2. Workers are heterogeneous in multi-dimensional skills.
  3. Frictional Labor Market $\rightarrow$ Workforce is a state variable for the firms.
     - Directed Search from Unemployment.
- Purpose of the Model
  1. Decompose impact of population aging on productivity into multiple-channels
     - Technology Adoption.
     - Firm Dynamics + Allocation of workers to firms via search frictions.
  2. Policy Counterfactuals: e.g., subsidy to on-the-job training / changing firing costs.

Agents

- **Workers**, heterogenous in
  - Tech skills, high (H) or low (L): $s_T \in \{\gamma_T^H, \gamma_T^L\}$
  - Production skills, high (H) or low (L): $s_P \in \{\gamma_P^H, \gamma_P^L\}$
  - Four types: $s \in \mathcal{S} \equiv \{LL, LH, HL, HH\}$
- **Multi-worker firms**, heterogeneous in
  - Workforce Composition: $\vec{n} = \{n_s\}_{s \in \mathcal{S}}$
  - Idiosyncratic Productivity (z): Exogenous
  - Technological Level (τ): Endogenous

Demographics and Skill Transitions

Workers' skill transitions are inherently linked to the individual life cycle:

- New entrants ("young"):
  - start with high tech skills ($s_T = H$) with probability ω
  - start with low production skills ($s_P = L$).
- Over time:
  - Tech skills become obsolete at rate λ_T: $\gamma_T^H \to \gamma_T^L$
  - Workers accumulate production experience at rate λ_P: $\gamma_P^L \to \gamma_P^H$
- Firms can re-train workers in tech skills at cost c_e: $\gamma_T^L \to \gamma_T^H$
- Workers retire at rate λ_r.

Population $\mathcal{L}$ evolves according to $\underbrace{\mu_t}_{\text{New Entrants}} = \mathcal{L}_t \underbrace{g}_{\text{Population growth rate}} + \lambda_r \mathcal{L}_t$

Production Technologies

- At any instant, two technologies are available:
  - Frontier ($\tau = 0$) and Laggard ($\tau = 1$)
- The production function of technology τ is

$$y_\tau \; = \; \underbrace{z\lambda^{-\tau}}_{\text{TFP}} \; \underbrace{\left[\alpha_\tau L_T^\rho + (1-\alpha_\tau) L_P^\rho\right]^{\frac{\eta}{\rho}}}_{\text{Labor Composite}}, \quad \eta < 1, \;\; \lambda > 1$$

- **Trade-off**
  - Frontier ($\tau = 0$): higher TFP but load more on high-skill workers ($\alpha_0 > \alpha_1$)

Task Assignment

Given $\vec{n}$, workers are split between Tech (T) and Production (P) to solve

$$y_\tau^* = \max_{\vec{n}} \; y_\tau(L_T, L_P)$$

$$\text{s.t.} \quad L_T = \gamma_T^H \underbrace{\left(n_{HH}^T + n_{HL}^T\right)}_{\text{High Skill}} + \gamma_T^L \underbrace{\left(n_{LL}^T + n_{LH}^T\right)}_{\text{Low Skill}} \qquad \text{(Tech Workers)}$$

$$L_P = \gamma_P^H \underbrace{(n_{HH}^P + n_{LH}^P)}_{\text{High Skill}} + \gamma_P^L \underbrace{(n_{LL}^P + n_{HL}^P)}_{\text{Low Skill}} \qquad \text{(Production Workers)}$$

Technology Transitions

Firms can pay $\mathbf{a} \geq 0$ to affect the following:

- **Tech-Upgrade**. Laggard become Frontier at rate

$$q_{10}(\underset{+}{\mathbf{a}}) \equiv \mathbb{P}(\text{Upgrade}) = \bar{q}_{10}[1 - \exp(-\chi \mathbf{a})]$$

- **Tech-Downgrade**. Frontier become Laggard at rate

$$q_{01}(\underset{-}{\mathbf{a}}) \equiv \mathbb{P}(\text{Downgrade}) = \bar{q}_{01} \exp(-\varphi \mathbf{a})$$

- **Obsolescence**. Laggard firms exit the market at rate

$$q_{12}(\underset{-}{\mathbf{a}}) \equiv \mathbb{P}(\text{Obsolescence}) = \bar{q}_{12} \exp(-\varphi \mathbf{a})$$

Unemployed Workers

The value of an unemployed worker of type $i \in \mathcal{S}$ is

$$(\rho+\lambda^r)\mathcal{U}_i = \max_W \Big\{ b + \underbrace{\mu_i(W)}_{\text{Job Finding Rate}} \max\{W - \mathcal{U}_i, 0\} + \underbrace{\lambda_T(\mathcal{U}_{L,s_P} - \mathcal{U}_{H,s_P})}_{\text{Tech Obsolescence}} \underbrace{\mathbb{1}\{s_T = H\}}_{\text{High-tech Worker}} \Big\}$$

$$\implies \mu_i(W) = \frac{(\rho+\lambda^r)\mathcal{U}_i - b - \lambda_T(\mathcal{U}_{L,s_P} - \mathcal{U}_{H,s_P})\mathbb{1}\{s_T = H\}}{W - \mathcal{U}_i}, \qquad s_P \in \{L, H\}$$

Trade-off: job-finding rate is **inversely related** to the contract's promised value W

- $W \uparrow \implies$ submarket (i, W) attracts more workers $\implies \mu_i(W) \downarrow$, wait time $\uparrow$
- Continuum of active submarkets maximizing unemployment value.

Firms

- State Variables: $\vec{s} \equiv (z, \tau, \vec{n}, \vec{W}) = (z, \tau, \{n_i, W_i\}_{i \in \mathcal{S}}), \quad \vec{n} \in \mathcal{N}_+^4$
- Hiring Process: firms post **one vacancy for each skill type** $i \in \mathcal{S}$
  - Choose the sub-markets $\{W_i\}_{i \in \mathcal{S}}$, taking job filling rate as given
  - Choose recruitment effort $\{v_i\}_{i \in \mathcal{S}}$ to increase matching probability, at cost $C_{v_i}(v_i)$
- Dynamic contracts $\mathcal{C}^{\Sigma}(\vec{s})$:

$$\mathcal{C}^{\Sigma} = \left\{ \{w_i,\ v_i,\ \delta_i,\ W_i'(s_i^+)\}_{i \in \mathcal{S}},\ \ \pi_e,\ \ a_\tau \right\}$$

  - Wage: $w_i, \ \forall i$
  - Recruitment intensity: $v_i, \forall i$, at cost $C_{v_i}(v_i)$
  - Firing rate: $\delta_i, \forall i$ at cost $n_i C_{\delta_i}(\delta_i)$
  - Continuation Promise: $W_i'(s_i^+)$
  - Up-Skilling rate: π_e at cost $n_{L,s_P} C_e(\pi_e)$
  - Investment: a_τ at cost $C_\tau(a)$
- **Exogenous Shocks**

Key Property of the Problem

- Assume:
    1. Utilities are linear and transferrable across agents
    2. Contract space is complete: state-contingent contracts
    3. Directed search
- Define the **Joint Surplus**: $\Sigma(\overrightarrow{n}, z, \tau) \equiv \underbrace{J(\overrightarrow{s})}_{\text{Firm's Value}} + \sum_i n_i \underbrace{W_i}_{\text{Worker's Value}}$

Proposition (Schaal [2017])

Solving the firm's problem is equivalent to maximize to joint surplus over the simpler contract space:

$$\mathcal{C}^{\Sigma} = \Big\{ \{ \underbrace{v_i}_{\text{Recruitment}}, \underbrace{\delta_i}_{\text{Firing}}, \underbrace{W_i'(s_i^+)}_{\text{Continuation Promise}} \}_{i \in \mathcal{S}}, \underbrace{\pi_e}_{\text{Re-training}}, \underbrace{a_T}_{\text{Tech Investment}} \Big\}$$

The joint surplus doesn't depend on promised utilities: new state space: $s = (\vec{n}, z, \tau)$

Joint Surplus Maximization

$$(\rho+s^F)\Sigma(s) = \max_{\mathcal{C}^\Sigma}\Bigg\{ y - C_e(\pi_e)(n_{LL}+n_{LH}) - C_\tau(a_\tau) - \sum_i \Big[C_{\delta_i}(\delta_i)n_i + C_{v_i}(v_i)\Big]$$

Outside Option $$+\sum_i n_i\left(\delta_i + s_i^W + s^F\right)\mathcal{U}_i$$

Utility to New Hires $$-\sum_i v_i\eta_i\left(W_i'(s_i^+)\right)W_i'(s_i^+)$$

Hiring $$+\sum_i v_i\eta_i\left(W_i'(s_i^+)\right)\left(\Sigma(s_i^+)-\Sigma(s)\right)$$

Separations $$+\sum_i\left(\delta_i+s_i^W\right)n_i\left(\Sigma(s_i^-)-\Sigma(s)\right) + \sum_i \lambda_r n_i\left(\Sigma(\vec{s}_{\lambda_o})-\Sigma(\vec{s})\right)$$

Skill Change $$+\sum_{i\in\{LL,LH\}}\pi_e\left(\Sigma(\vec{s}_{\pi_e})-\Sigma(\vec{s})\right)+\sum_{i\in\{HL,HH\}}\lambda_T\left(\Sigma(\vec{s}_{\lambda_T})-\Sigma(\vec{s})\right)+\sum_{i\in\{LL,HL\}}\lambda_P\left(\Sigma(\vec{s}_{\lambda_P})-\Sigma(\vec{s})\right)$$

Tech Change $$+\sum_{\tau'}\pi(\tau'|\tau.a_\tau)(\Sigma(s_{\tau'})-\Sigma(s))$$

Prod. Shock $$+\sum_{z'}F_z(z'|z)(\Sigma(s_{z'})-J(s))\Bigg\}$$

Workers' Continuation Promises

- Assume Cobb-Douglas matching function with **matching elasticity** γ,

$$M(V,U) = V^{\gamma} U^{1-\gamma}.$$

- Then, worker's continuation promise is

$$W_i'(s_i^+) \;=\; \gamma \mathcal{U}_i \;+\; (1-\gamma) \underbrace{\left[\Sigma(s_i^+) - \Sigma(s)\right]}_{\text{Surplus Gain from Hiring}} \;, \;\; \forall i \in \mathcal{S},$$

equivalent to Nash bargaining solution evaluated at the **Hosios condition**.

Potential Entrants

- Potential entrants' value:

$$J^e = \underbrace{-\kappa}_{\text{Entry Cost}} + \underbrace{p_0 \tilde{J}^e(\tau = 0) + (1 - p_0)\tilde{J}^e(\tau = 1)}_{\text{Expected Profits}}$$

  - Pay a flow cost to open a business: κ
  - Enter as frontier with probability p_0.

- Upon entry:

$$\tilde{J}^e(\tau) = \sum_z F^e(z) \left[\max_{\{W_i'\}} \sum_{i \in \mathcal{S}} \eta_i \left(W_i'(s_i^e) \right) J(s_i^e) \right], \ \forall i$$

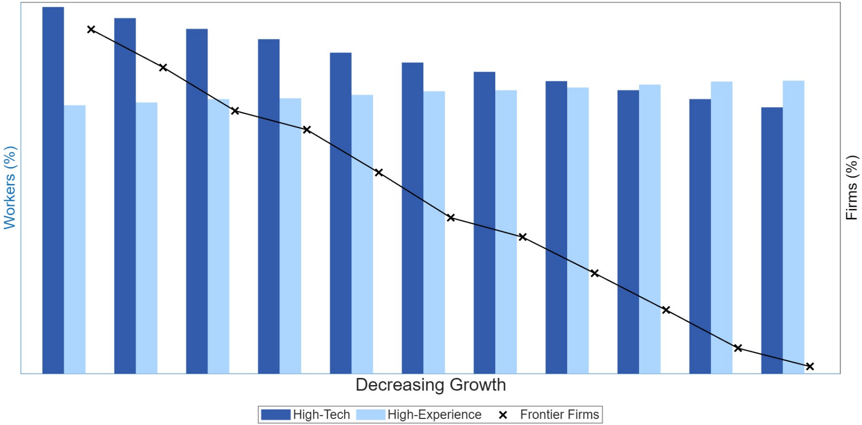
  - Draw a **random productivity shock** from F_z^e
  - **Post one vacancy** for each type of worker, choosing the submarket W_i.

Stationary Equilibrium

A stationary equilibrium consists of **value functions** $J(s), W(s), \Sigma(s)$, **policy functions** $\left\{w_i(s), v_i(s), \delta_i(s), W_i'(s)\right\}_{i\in\mathcal{S}}, \pi_e(s), a_\tau(s)$, **unemployment values** $\{U_i\}_{i\in\mathcal{S}}$, a **per-capita distribution** $\Lambda(s)$, **per-capita masses of active firms** F and **entrants** F_e, **per capita measures of employment and unemployment** per each type $\{E_i, U_i\}_{i\in\mathcal{S}}$, such that:

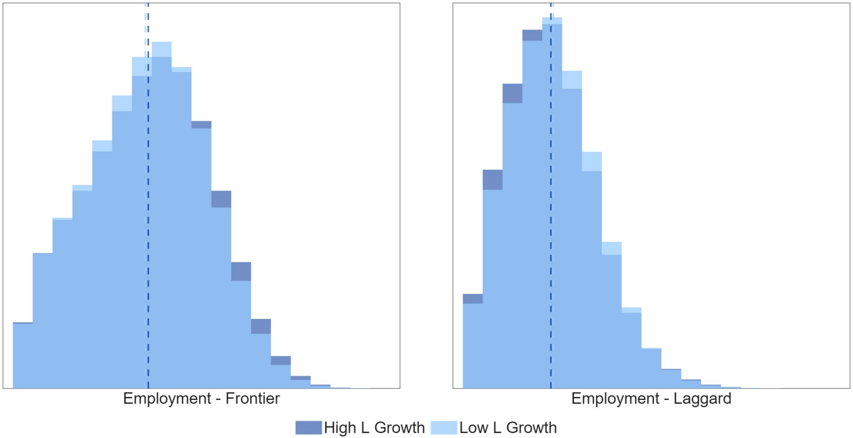
- ***Optimality holds***.
- ***Distributions and measures per capita are stationary***.
- ***Free-entry holds***: $J^e = 0$

Skill Composition and Technology Adoption



- Low g $\implies$ $\uparrow$ Experience-Skills, $\downarrow$ Tech-Skills, $\downarrow$ Frontier Adoption.

Employment Distribution Polarization



- Low $g \implies$ Frontier become smaller + Laggard become bigger.

Outline

Aggregate Productivity

Let **productivity** be defined as output per-worker $(\mathcal{Y}/E)$.

Productivity Decomposition

Aggregate productivity can be decomposed as

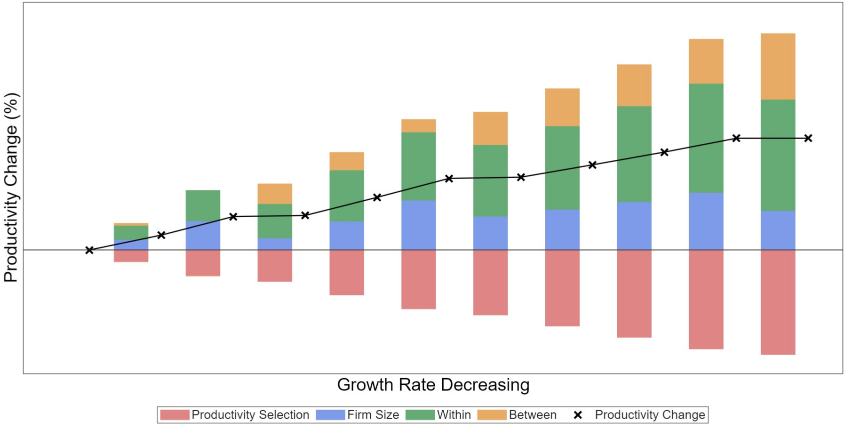
$$\text{Productivity} = \underbrace{(\text{Avg Size})^{1-\eta}}_{\text{DRS}} \times \text{Avg (Standardized) Output} =$$

$$= (\text{Avg Size})^{1-\eta} \times \underbrace{\sum_{z,\tau} \text{Share}(z,\tau)}_{\text{Productivity Selection}} \times \text{Avg (Standardized) Output}(z,\tau)$$

$$= (\text{Avg Size})^{1-\eta} \times \sum_{z,\tau} \text{Share}(z,\tau) \times \sum_{\substack{\text{Scale} \\ n}} \sum_{\substack{\text{Composition} \\ m}} \text{Output}(n,m) \times \underbrace{\text{Share}(m|n)}_{\substack{\text{Within-firm} \\ \text{Misallocation}}} \times \underbrace{\text{Share}(n)}_{\substack{\text{Between-firm} \\ \text{Misallocation}}}$$

Derivation

Productivity Decomposition



- Low $g \implies \uparrow$ Loss from Productivity Selection.

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Notation: Evolution of State Variables

Back

Remark: Given continuous time, only one state transition can happen simultaneously.

$$\vec{s}' \equiv (\vec{n}, z', \tau') \in \begin{cases} (n_{LL}+1, \vec{s}_-),\ (n_{LH}+1, \vec{s}_-),\ (n_{HL}+1, \vec{s}_-),\ (n_{HH}+1, \vec{s}_-) & \text{Hiring } i: \ \vec{s}_i^{\,+},\ \forall i \\ (n_{LL}-1, \vec{s}_-),\ (n_{LH}-1, \vec{s}_-),\ (n_{HL}-1, \vec{s}_-),\ (n_{HH}-1, \vec{s}_-) & \text{Firing } i: \ \vec{s}_i^{\,-},\ \forall i \\ (n_{LL}-1, n_{HL}+1, \vec{s}_-),\ (n_{LH}-1, n_{HH}+1, \vec{s}_-) & \text{Up-skilling}: \ \vec{s}_{\pi_e} \\ (n_{HL}-1, n_{LL}+1, \vec{s}_-),\ (n_{HH}-1, n_{LH}+1, \vec{s}_-) & \text{Obsolescence}: \ \vec{s}_{\lambda_T} \\ (n_{LL}-1, n_{LH}+1, \vec{s}_-),\ (n_{HL}-1, n_{HH}+1, \vec{s}_-) & \text{Experience}: \ \vec{s}_{\lambda_R} \\ (n_{LL}-1, \vec{s}_-),\ (n_{LH}-1, \vec{s}_-),\ (n_{HL}-1, \vec{s}_-),\ (n_{HH}-1, \vec{s}_-) & \text{Retirement}: \ \vec{s}_{\lambda_o} \\ (\tau', \vec{s}_-) & \text{Tech Change}: \ \vec{s}_{\tau'} \\ (z', \vec{s}_-) & \text{Prod. Shock}: \ \vec{s}_{z'} \end{cases}$$

Firm's Problem

Back

- Long-Term Contract: $\vec{\mathcal{C}} = \left\{ \{ w_i, v_i, \delta_i, W_i'(\vec{s})' \}_{i \in \{LL,LH,HL,HH\}},\ \pi_e,\ a_\tau \right\}$

$$\rho J(\vec{s}) = \max_{\vec{\mathcal{C}}} \Bigg\{ y - C_e(\pi_e)(n_{LL} + n_{LH}) - C_\tau(a_\tau) + s^F \left(J^e - J(\vec{s}) \right)$$

$$\textit{Workforce Costs} \quad - \sum_i \left[w_i n_i + C_{\delta_i}(\delta_i) n_i + C_{v_i}(v_i) \right]$$

$$\textit{Hiring} \quad + \sum_i v_i \eta_i \left(W_i'(\vec{s}_i^{\,+}) \right) \left(J(\vec{s}_i^{\,+}) - J(\vec{s}) \right) \Bigg\}$$

$$\textit{Separations} \quad + \sum_i \left(\delta_i + s_i^W \right) n_i \left(J(\vec{s}_i^{\,-}) - J(\vec{s}) \right) + \sum_i \lambda_r n_i \left(J(\vec{s}_{\lambda_o}) - J(\vec{s}) \right)$$

$$\textit{Skill Change} \quad + \sum_{i \in \{LL,LH\}} \pi_e \left(J(\vec{s}_{\pi_e}) - J(\vec{s}) \right) + \sum_{i \in \{HL,HH\}} \lambda_T \left(J(\vec{s}_{\lambda_T}) - J(\vec{s}) \right) + \sum_{i \in \{LL,HL\}} \lambda_P \left(J(\vec{s}_{\lambda_P}) - J(\vec{s}) \right)$$

$$\textit{Tech Change} \quad + \sum_{\tau'} \pi(\tau' | \tau, a) \left(J(\vec{s}_{\tau'}) - J(\vec{s}) \right)$$

$$\textit{Prod. Shock} \quad + \sum_{z'} F_z(z'|z) \left(J(\vec{s}_{z'}) - J(\vec{s}) \right) \Bigg\} \qquad \text{s.t.} \quad \mathcal{W}_i(\vec{\mathcal{C}}) \geq W_i, \quad W_i' \geq \mathcal{U}_i, \quad \forall i$$

Aggregate Productivity Decomposition

Back

Aggregate productivity

$$\frac{\mathcal{Y}}{E} = \frac{1}{E}\sum_{s\in\mathcal{S}} y^*(s)f(s)$$
$$= \frac{F}{E}\sum_{z,\tau}\frac{F_{z,\tau}}{F}\sum_{n} y^*(s)\tilde{f}_{z,\tau}(n)$$
$$= \underbrace{\left(\frac{F}{E}\right)^{1-\nu}}_{\text{Firm Size}}\sum_{z,\tau}\underbrace{\frac{F_{z,\tau}}{F}}_{\text{Firm Selection}}\underbrace{\sum_{\hat{n},m} y^*_{z,\tau}(\hat{n},m)g_{z,\tau}(\hat{n},m)}_{\text{Workers Allocation}} \tag{1}$$

where

$$g_{z,\tau}(\hat{n},m) = \sum_{n}\tilde{f}_{z,\tau}(n)\mathbb{I}\left\{\frac{n}{E/F} = \hat{n} \;\; \& \;\; m_\tau = m\right\}$$